SUPPLEMENTARY READING: VANDER MONDE MATRIX

Question 1. [Vander Monde] Evaluate the determinant of the matrix

$$V_n = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}.$$

SOLUTIONS.

1. Add $(-x_1) \times (n-1)$-th column to the n-th column, and then add $(-x_1) \times (n-2)$-th column to the $(n-1)$-th column. Repeat this process till we add $(-x_1) \times$ 1st column to the 2nd column. It yields

$$V_n \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & x_2^2 - x_2 x_1 & \cdots & x_2^{n-1} - x_2^{n-2} x_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n - x_1 & x_n^2 - x_n x_1 & \cdots & x_n^{n-1} - x_n^{n-2} x_1 \end{bmatrix}$$

Therefore,

$$\det(V_n) = \prod_{i=2}^{n} (x_i - x_1) \det \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & x_2 & \cdots & x_2^{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & x_n & \cdots & x_n^{n-2} \end{bmatrix}$$

We denote by V_{n-1} the submatrix $\begin{bmatrix} 1 & x_2 & \cdots & x_2^{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} \end{bmatrix}$. Then by the properties of the determinants, it follows that $\det(V_n) = \prod_{i=2}^{n} (x_i - x_1) \det(V_{n-1})$. Repeat the trick with $-x_1$ replaced by $-x_2$. Then we get

$$\det(V_n) = \prod_{i=2}^{n} (x_i - x_1) \prod_{i=3}^{n} (x_i - x_2) \det(V_{n-2}).$$

Here V_{n-2} is the matrix $\begin{bmatrix} 1 & x_3 & \cdots & x_3^{n-3} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-3} \end{bmatrix}$. This recursive formula leads to the consequence

$$\det(V_n) = \prod_{1 \le i < j \le n} (x_j - x_i).$$